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Non-conformal renormalised stress tensors in Robertson–Walker space–times

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Abstract. We consider a number of explicitly soluble models of a massless, non-conformally-coupled scalar quantum field propagating in certain special Robertson–Walker background space–times.

Regularisation is effected using covariant point-splitting, and renormalised stress tensors are constructed according to the *ansatz*, described in our companion paper, based on the DeWitt–Schwinger expansion. The results are all finite, conserved and free of regularisation ambiguities. From these special cases we are able to deduce a great deal about the structure of renormalised stress tensors in more general models in which computation-approximation techniques will be necessary.

1. Introduction

In a companion paper (Bunch and Davies 1978, hereafter referred to as II) we treated a massive scalar field propagating in de Sitter space, and adapted our earlier work with Christensen and Fulling (Davies *et al* 1977, hereafter referred to as I) on point-splitting regularisation to this non-conformal case. Following suggestions by Christensen and Fulling (1977) we employed a renormalisation *ansatz* based on the DeWitt–Schwinger inverse mass expansion, and obtained a renormalised stress tensor in agreement with other authors (Dowker and Critchley 1976) who use dimensional regularisation.

In more general space–times, it is hard to find mode solutions of the massive wave equation, although there are a number of cases in which the massless, *minimally* coupled scalar wave equation possesses tractable solutions. As this situation is conformally non-trivial (the minimally coupled wave equation in four dimensions is not invariant under conformal transformations) interesting features, such as logarithmic terms, which were present in the massive de Sitter calculation, survive in these other cases. However, because we are dealing with a *massless* field, some subtle differences occur. By considering some explicitly soluble special cases, considerable light is cast on the structure of renormalised stress tensors in more general cases. All the cases treated here involve a Robertson–Walker background space–time, the conformal flatness of which, whilst of no direct assistance to the solution of the non-conformal wave equation under investigation, nevertheless saves a lot of labour, because we can use the expansions, expressions and experience of our earlier conformal treatment (I). It turns out that even in the non-conformal case, the point-split expressions all reduce

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to functions of the product of null separations $\Delta u\ \Delta v$, and the stress tensors all have the form of the conformal tensor found in I, plus non-conformal 'correction terms'. As the conformal symmetry breaking is associated physically with particle creation, the correction terms can, loosely speaking, be thought as representing the contribution to the stress tensor of this particle creation.

Another interesting feature to emerge is that the correction terms do not require any further renormalisation over and above the conformal term, except for logarithmic terms. However, the logarithmic terms can be deduced by direct inspection of the logarithmically divergent pieces of Christensen's (1976) DeWitt–Schwinger expansion for a general space–time (these divergent terms are independent of the quantum state) which turn out to have completely unambiguous geometrical coefficients. This means that it ought to be possible to write out the non-conformal scalar stress tensor for a general conformally flat space–time as a known (renormalised) geometrical tensor, plus a (generally) non-geometrical, non-local correction term, arranged as a *finite* additional mode sum, to be evaluated in general by some computation-approximation technique.

In § 2, as a preliminary, we give a two-dimensional concrete calculation, in which a massless scalar field ϕ, with an additional ξR term in the wave equation breaking the conformal symmetry, propagates in a certain Robertson–Walker background space–time. The two-point function $G(x'', x') = \langle\phi(x'')\phi(x')\rangle$, evaluated as an expectation value in a particular natural vacuum state, is displayed explicitly and differentiated to yield an unrenormalised stress tensor. Renormalisation is then effected by invoking our *ansatz* explained in detail in II. The final renormalised stress tensor is (automatically) finite, conserved, free of regularisation ambiguities and, by accident, purely local and geometrical in form.

In § 3, the treatment is extended to two special four-dimensional Robertson–Walker space–times, although for convenience we choose to renormalise G first, rather than work with the full stress tensor. Once again, the results are accidently local (i.e., pseudo-local in the terminology of I), although in one case non-geometrical.

Section 4 is a discussion of the way in which the results of these special examples may be used to deduce the structure of renormalised stress tensors in more general situations.

2. Two-dimensional case

As a first illustration, consider a massless scalar field ϕ which satisfies the wave equation

$$(\Box + \xi R)\phi = 0 \tag{2.1}$$

in two-dimensional space–time, where ξ is a parameter and R is the curvature scalar. The field is conformally coupled if $\xi = 0$, and this case has been treated for a general background space–time by Davies and Fulling (1977) for the scalar field and Davies and Unruh (1977) for the neutrino field. Here we consider the case of a general ξ. Under the circumstances $\xi \neq 0$, particle production will occur if the space–time is not static. We treat here the special case of a Robertson–Walker space–time, so that the scalar curvature R is a function of time only.

It is convenient to work with the conformally flat form of the metric

$$ds^2 = C(\eta)(d\eta^2 - dz^2) \tag{2.2}$$

for which the wave equation (2.1) may be solved in terms of mode solutions of the form $\psi_k(\eta)e^{ikz}$, where $\psi_k(\eta)$ satisfies the equation

$$\frac{d^2\psi_k}{d\eta^2}+(k^2+\xi RC)\psi_k=0. \tag{2.3}$$

Exact solutions of equation (2.3) are hard to find, but if the function $C(\eta)$ is chosen to be of a special form, it is possible to obtain tractable modes ψ_k. One example which is immediately suggested by the form of equation (2.3) is

$$\xi RC=\beta^2=\text{constant} \tag{2.4}$$

which reduces (2.3) to the form of the *massive* scalar wave equation in *flat* space–time with the constant β playing the role of the mass. Using the definition of the scalar curvature in two-dimensional space–time

$$R=4\left(\frac{C_{uv}}{C^2}-\frac{C_uC_v}{C^3}\right) \tag{2.5}$$

where $u\equiv\eta-z, v\equiv\eta+z$ are standard retarded and advanced null coordinates respectively, and $C_u\equiv\partial C/\partial u$ etc, we obtain from (2.4) the conformal scale factor for this particular Robertson–Walker universe

$$C(\eta)\propto\exp\left[A\eta+(\beta^2/2\xi)\eta^2\right] \tag{2.6}$$

where A is a constant. Choosing $\beta=0$ (hence $R=0$) yields the two-dimensional Milne universe if $A\neq0$. This is just a portion of Minkowski space in disguise.

The solution of equation (2.3) with the restriction (2.4) is immediate

$$\psi_k(\eta)\propto e^{-i\omega\eta}, \qquad \omega^2=k^2+\beta^2 \tag{2.7}$$

whence the normalised positive frequency mode solutions of (2.1) are simply

$$(4\pi\omega)^{-1/2}e^{i(kz-\omega\eta)}. \tag{2.8}$$

First we calculate the two-point function

$$G(x'',x')\equiv\langle\phi(x'')\phi(x')\rangle \tag{2.9}$$

where x denotes a space–time point (η, z) and $\langle\ \rangle$ refers to the expectation value in the vacuum state associated with the modes (2.8). It is understood throughout that symmetrisation over x'', x' is to be performed if necessary. $G(x'',x')$ may be obtained in the usual way as a mode sum

$$G(x'',x')=(4\pi)^{-1}\int_{-\infty}^{\infty}\omega^{-1}\,e^{ik\Delta z-i\omega\Delta\eta}\,dk \tag{2.10}$$

where $\Delta\eta=\eta''-\eta'$, $\Delta z=z''-z'$. The integral in (2.10) may be performed in terms of a MacDonald function

$$G(x'',x')=\frac{1}{2\pi}K_0[i\beta(\Delta u\ \Delta v)^{1/2}] \tag{2.11}$$

with $\Delta u=\Delta\eta-\Delta z$, $\Delta v=\Delta\eta+\Delta z$.

We are only interested in the behaviour of G as Δu, $\Delta v\to0$, so we expand (2.11) in powers of the argument (real part understood)

$$2\pi G(x'',x')=-\gamma-\tfrac{1}{2}\ln(\tfrac{1}{4}\beta^2\,\Delta u\,\Delta v)-\tfrac{1}{4}\beta^2\Delta u\,\Delta v[-\gamma-\tfrac{1}{2}\ln(\tfrac{1}{4}\beta^2\Delta u\,\Delta v)+1]+\ldots \tag{2.12}$$

from which it may be seen that G diverges logarithmically as $x'' \to x'$. This is the usual ultra-violet divergence. Also note that G diverges logarithmically at the lower limit as $\beta \to 0$. This is an infra-red divergence. It is interesting that the infra-red divergence is present in this model only when the coupling of the scalar field is conformal.

Next, we calculate the stress tensor expectation value by differentiation of G according to the formal relation (Christensen 1976)

$$\langle T_{\mu\nu}(x)\rangle = \lim_{x'',x'\to x} [\tfrac{1}{2}(1-2\xi)(G_{;\mu'\nu''}+G_{;\mu''\nu'})-\xi(G_{;\mu''\nu''}+G_{;\mu'\nu'})+(2\xi-\tfrac{1}{2})g_{\mu\nu}G_{;\sigma''}{}^{\sigma'}$$
$$+\xi g_{\mu\nu}(G_{;\sigma''}{}^{\sigma''}+G_{;\sigma'}{}^{\sigma'})-\tfrac{1}{2}g_{\mu\nu}m^2G] \tag{2.13}$$

where the primes on the indices indicate at which point the differentiation is to be performed. To give (2.13) a covariant meaning, one chooses x to lie on the midpoint of a geodesic joining x' and x'', at a proper distance ϵ from each. Then in order to perform the differentiation symbolised in (2.13) it is necessary to parallel transport the derivatives from x' and x'' back to x. This differentiation procedure, while complicated and laborious, is now routine in these regularisation calculations, and can readily be performed using the contents of the appendices in I and II.

First it is necessary to rewrite (2.12) in terms of ϵ and t^σ, the tangent vector to the geodesic at x, by expanding Δu and Δv in a power series in ϵ (see Davies and Fulling 1977):

$$G(x'',x') = -(2\pi)^{-1}\{(1-\epsilon^2\Sigma\xi R)[\gamma+\tfrac{1}{2}\ln(\epsilon^2\Sigma\xi R)]+\epsilon^2\Sigma(A_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1}-\xi R)\} \tag{2.14}$$

where $\Sigma = t^\sigma t_\sigma = \pm 1$ and $A_{\alpha\beta}$ is a tensor with null components

$$\begin{aligned} A_{uu} &= A_{vv} = \tfrac{1}{12}(-\dot{D}+2D^2) \\ A_{uv} &= A_{vu} = -\tfrac{1}{6}\dot{D} \end{aligned} \tag{2.15}$$

and $D = \dot{C}/C$, the dot denoting differentiation with respect to u or v ($\partial/\partial u = \partial/\partial v = \tfrac{1}{2}\partial/\partial\eta$ when $C = C(\eta)$).

The differentiation is now performed using equations (D.7a, b) of I and equations (C.1) and (C.5) of II. After some work we find, using (2.13) and (2.14)

$$\langle T_{\mu\nu}(x)\rangle = (8\pi)^{-1}[-(\Sigma\epsilon^2)^{-1}+(\tfrac{1}{6}-\xi)R](g_{\mu\nu}-2\Sigma^{-1}t_\mu t_\nu)$$
$$+\theta_{\mu\nu}+\frac{\xi}{4\pi}[R^{-1}R_{;\mu\nu}-R^{-2}R_{;\mu}R_{;\nu}-g_{\mu\nu}(R^{-1}\Box R-R^{-2}R^{;\sigma}R_{;\sigma})] \tag{2.16}$$

where the traceless tensor $\theta_{\mu\nu}$ has components

$$\begin{aligned} \theta_{uu} &= (24\pi)^{-1}(C^{-1}C_{uu}-\tfrac{3}{2}C^{-2}C_u^2) \\ \theta_{vv} &= (24\pi)^{-1}(C^{-1}C_{vv}-\tfrac{3}{2}C^{-2}C_v^2) \\ \theta_{uv} &= \theta_{vu} = 0. \end{aligned} \tag{2.17}$$

Comparison of (2.16) with equation (2.3) of Davies and Fulling (1977) shows that the answer is identical (to within sign conventions), when $\xi = 0$. Notice that there is no logarithmic divergence in $\langle T_{\mu\nu}\rangle$ in spite of the fact that a term of the form $\epsilon^2 \ln \epsilon^2$ appears in G: exact cancellation occurs between the different terms in (2.13) which go to make up $\langle T_{\mu\nu}\rangle$.

To obtain the renormalised stress tensor from (2.16) it is necessary to subtract the terms of the DeWitt–Schwinger expansion up to adiabatic order R, which are (see II and Christensen and Fulling 1977), in the massless limit

$$-(8\pi\epsilon^2\Sigma)^{-1}(g_{\mu\nu}-2\Sigma^{-1}t_\mu t_\nu)-(4\pi\Sigma)^{-1}(\tfrac{1}{6}-\xi)Rt_\mu t_\nu. \tag{2.18}$$

Alternatively, G could have been renormalised by such a subtraction (see equation (4.5) of II), and then differentiated to obtain the renormalised stress tensor $\langle T_{\mu\nu}\rangle_{\text{ren}}$ directly. Either way we find

$$\langle T_{\mu\nu}\rangle_{\text{ren}}=\theta_{\mu\nu}+\frac{1}{48\pi}Rg_{\mu\nu}+\xi\left(\Lambda_{\mu\nu}+\frac{1}{8\pi}Rg_{\mu\nu}\right) \tag{2.19}$$

$$\Lambda_{\mu\nu}=(4\pi)^{-1}[R^{-1}R_{;\mu\nu}-R^{-2}R_{;\mu}R_{;\nu}-g_{\mu\nu}(R^{-1}\Box R-R^{-2}R^{;\sigma}R_{;\sigma})] \tag{2.20}$$

$$\Lambda^\sigma{}_\sigma=(4\pi)^{-1}[R^{-2}R^{;\sigma}R_{;\sigma}-R^{-1}\Box R]. \tag{2.21}$$

The tensor $\Lambda_{\mu\nu}$ represents the conformal-breaking contribution to the stress tensor. Physically, the presence of this term is associated with particle production by the expanding space. The trace of the renormalised stress tensor is

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle_{\text{ren}}=\frac{1}{8\pi}\left(\frac{1}{6}-\xi\right)R+\frac{\xi}{4\pi}(R^{-2}R^{;\sigma}R_{;\sigma}-R^{-1}\Box R). \tag{2.22}$$

The first term on the right-hand side of (2.22) can be traced back to the last term in equation (2.13), which gives a contribution to the renormalised stress tensor even in the massless limit, $m\to 0$, because of the presence of an m^{-2} term in the DeWitt-Schwinger expansion for G (see Christensen and Fulling 1977). Such a contribution is generally known as 'anomalous' because it breaks the conformal symmetry present in the operator $T_{\mu\nu}$. The last term in equation (2.22) is not anomalous: it is the renormalised expectation value of the trace of the massless stress tensor operator $T^\sigma{}_\sigma\equiv\xi\Box\phi^2$.

We know from Davies and Fulling (1977), and expect on general grounds, that the *conformal* stress tensor ($\langle T_{\mu\nu}\rangle$ with $\xi=0$) is conserved. Hence both conformal and non-conformal terms are separately conserved:

$$\theta^{\mu\nu}{}_{;\nu}=-\frac{1}{48\pi}R^{;\mu} \tag{2.23}$$

$$\Lambda^{\mu\nu}{}_{;\nu}=-\frac{1}{8\pi}R^{;\mu} \tag{2.24}$$

as may be verified by direct computation.

3. Four-dimensional cases

In four dimensions, a number of special conformally non-trivial cases may be solved exactly. We consider here two examples of *minimal* coupling for a massless scalar field propagating in a conformally flat (in fact Robertson–Walker) background space-time.

The wave equation is

$$\Box\phi=0 \tag{3.1}$$

and the metric is

$$ds^2 = C(\eta)(d\eta^2 - d\boldsymbol{x}.d\boldsymbol{x}) \tag{3.2}$$

where $C(\eta) = a^2(t)$, a being the usual Robertson–Walker scale factor, will take a specific functional form to enable (3.1) to be solved. Mode solutions of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\psi_k(\eta)$ with $k = |\boldsymbol{k}|$ may be chosen, where ψ_k satisfies

$$\frac{d}{d\eta}\left(C\frac{d\psi_k}{d\eta}\right) + Ck^2\psi_k = 0 \tag{3.3}$$

which is the analogue of equation (2.3) with $\xi = 0$.

Equation (3.3) may be rearranged as follows

$$\frac{d^2}{d\eta^2}(C^{1/2}\psi_k) + (k^2 - 6RC)C^{1/2}\psi_k = 0. \tag{3.4}$$

Consider the special case

$$a(t) = (1 - \beta^2 t^2)^{1/2} \tag{3.5}$$

or

$$C(\eta) = \cos^2 \beta\eta, \qquad \beta = \text{constant} \tag{3.6}$$

which corresponds to RC = constant. As in the two-dimensional example treated in the previous section, equation (3.4) resembles the massive wave equation in flat space–time for the field $C^{1/2}\psi_k$

$$\frac{d^2}{d\eta^2}(C^{1/2}\psi_k) + (k^2 + \beta^2)C^{1/2}\psi_k = 0 \tag{3.7}$$

with the constant β once again playing the role of the mass. Because of this formal similarity, we can write down mode solutions of (3.1) immediately for this special case, in the form of familiar exponentials

$$\psi_k \propto e^{i\boldsymbol{k}.\boldsymbol{x} - i\omega\eta}, \qquad \omega = (k^2 + \beta^2)^{1/2}. \tag{3.8}$$

Similarly the two-point function $G(x'', x') \equiv \langle\phi(x'')\phi(x')\rangle$ constructed as an expectation value in the vacuum state associated with the modes (3.8), is already known formally:

$$G(x'', x') = -\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{4\pi^2\,\Delta u\,\Delta v}\,i\beta K_1[i\beta(\Delta u\,\Delta v)^{1/2}] \tag{3.9}$$

where K_1 is a MacDonald function and we use retarded and advanced null coordinates $u \equiv \eta - z$, $v \equiv \eta + z$, choosing the point separation between x'' and x' to lie in the $\eta - z$ plane, without loss of generality (the space is isotropic). The factors $C^{-1/2}$ in (3.9) come from the fact that it is $C^{1/2}\psi_k$ rather than ψ_k which appears in equation (3.7).

It is interesting to note that the limit $\beta \to 0$ corresponds to the universe $a(t) \propto t^{1/2}$ (radiation-filled Friedmann model). In this case, $i\beta K_1(i\beta\sqrt{\Delta u\,\Delta v}) \to 1$ and G reduces to the *conformal* two-point function found in I. This universe has the property $R = 0$, and in that case the minimally- and conformally-coupled wave equations coincide. (In all cases, we find that G is a *sum* of the conformal part plus a conformal symmetry-breaking piece. The latter must always vanish when $R = m = 0$.)

The now familiar procedure of expanding the special function in $G(x'', x')$ and then expanding the Δu, Δv, η'', η' factors in powers of ϵ and t^σ corresponding to geodesics through the point x of interest, is carried out. We omit the intermediate details, as these follow along the lines of I and II. The result is (real part understood)

$$
\begin{aligned}
G(x'', x') = &-\frac{1}{16\pi^2\epsilon^2\Sigma} + \frac{1}{48\pi^2}(R_{\alpha\beta}t^\alpha t^\beta + \tfrac{1}{3}R) - \frac{R}{48\pi^2}\left[\gamma + \frac{1}{2}\ln\left(\frac{R\epsilon^2\Sigma}{6}\right)\right] \\
&+ \frac{\epsilon^2\Sigma}{2880\pi^2}(6R_{\alpha\beta;\gamma\delta}t^\alpha t^\beta t^\gamma t^\delta \Sigma^{-2} - R_{;\alpha\beta}t^\alpha t^\beta \Sigma^{-1} - 14R_{\alpha\beta}R_{\gamma\delta}t^\alpha t^\beta t^\gamma t^\delta \Sigma^{-2} \\
&+ 2R_\alpha{}^\sigma R_{\sigma\beta}t^\alpha t^\beta \Sigma^{-1} - 6RR_{\alpha\beta}t^\alpha t^\beta \Sigma^{-1} + \tfrac{91}{12}R^2) \\
&+ \frac{\epsilon^2\Sigma}{576\pi^2}\left[\gamma + \frac{1}{2}\ln\left(\frac{R\epsilon^2\Sigma}{6}\right)\right](4RR_{\alpha\beta}t^\alpha t^\beta \Sigma^{-1} - 2R_{;\alpha\beta}t^\alpha t^\beta \Sigma^{-1} \\
&+ 2\Box R - R^2) + O(\epsilon^4)
\end{aligned}
\tag{3.10}
$$

which may be written

$$
\begin{aligned}
G(x'', x') = &\, G_{\text{conformal}} + (576\pi^2)^{-1}\left[\gamma + \frac{1}{2}\ln\left(\frac{R\epsilon^2\Sigma}{6}\right)\right] \\
&\times[-12R + \epsilon^2\Sigma(4RR_{\alpha\beta}t^\alpha t^\beta \Sigma^{-1} - 2R_{;\alpha\beta}t^\alpha t^\beta \Sigma^{-1} + 2\Box R - R^2)] \\
&+ \frac{R}{96\pi^2} - \frac{R\epsilon^2\Sigma}{288\pi^2}[R_{;\alpha\beta}t^\alpha t^\beta - \tfrac{19}{24}R] + O(\epsilon^4)
\end{aligned}
\tag{3.11}
$$

where $G_{\text{conformal}}$ is the expression for $\langle\phi^2\rangle$ given for the conformally-coupled general Robertson–Walker case in I.

Equation (3.11) has a number of interesting features. Firstly, it is completely geometrical in the sense that all terms are functions of the Ricci tensor $R_{\alpha\beta}$, or R and its derivatives. (We include the logarithmic term in this definition of geometrical.) This feature turns out to be an accident of the particular choice (3.5) or (3.6) (see the discussion in § 4). Secondly, the structure of (3.11) is that of a conformal part plus a symmetry-breaking correction term, a feature which seems to be quite general, at least for conformally flat space–times. Because there is no distinction between conformal and minimal coupling if $R = 0$, the correction term vanishes when $R = 0$, as will be seen on inspection of (3.11). Thirdly, all the correction terms will lead to terms in $\langle T_{\mu\nu}\rangle$ which are *independent* of t^σ (this is always true of terms of the type $A_{\alpha\beta}t^\alpha t^\beta + B$, for some tensor $A_{\alpha\beta}$ and scalar B). Consequently, apart from easily-handled logarithmic terms, there is *no need* to carry out further regularisation of the non-conformal stress tensor once the conformal tensor has been regularised. The correction terms, to within a logarithmic term, are finite and unambiguous, another feature which seems to be quite general. As a check on the correctness of (3.10) we have verified by direct computation that it is a solution of the wave equation (3.1).

Renormalisation of (3.10) is now effected by subtracting the DeWitt–Schwinger terms up to adiabatic order R^2 (see II, equation (4.4)) to yield

$$
\begin{aligned}
G_{\text{ren}} = &\,(2880\pi^2)^{-1}\{20R - m^{-2}(R_{\alpha\beta}R^{\alpha\beta} - 6\Box R + \tfrac{13}{6}R^2) - 30R\ln(\tfrac{1}{6}Rm^{-2}) \\
&+ \epsilon^2\Sigma[R_{\alpha\beta}R^{\alpha\beta} + 6\Box R + \tfrac{19}{4}R^2 - (R_{;\alpha\beta} + 2R_\alpha{}^\rho R_{\rho\beta} + \tfrac{14}{3}RR_{\alpha\beta})t^\alpha t^\beta \Sigma^{-1} \\
&+ (10RR_{\alpha\beta}t^\alpha t^\beta \Sigma^{-1} - 5R_{;\alpha\beta}t^\alpha t^\beta \Sigma^{-1} + 5\Box R - \tfrac{5}{2}R^2)\ln(\tfrac{1}{6}Rm^{-2})]\} + O(m^2).
\end{aligned}
\tag{3.12}
$$

This quantity may now be differentiated to obtain $\langle T_{\mu\nu}\rangle_{\rm ren}$. This need not be done directly, because the result for a general expression has already been given in the appendixes of I and II. Substituting (3.11) into those expressions gives the answer

$$\langle T_{\mu\nu}\rangle_{\rm ren} = -\frac{^{(1)}H_{\mu\nu}}{1152\pi^2}\ln R\mu^{-2} + \frac{1}{69120\pi^2}(-168R_{;\mu\nu} + 288\Box R g_{\mu\nu} + 24R_{\mu\sigma}R^{\sigma}{}_{\nu} - 12R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu} - 64RR_{\mu\nu} + 63R^2 g_{\mu\nu}), \tag{3.13}$$

where the conserved tensor $^{(1)}H_{\mu\nu}$ is given by

$$^{(1)}H_{\mu\nu} = 2R_{;\mu\nu} - 2\Box R g_{\mu\nu} + 2(RR_{\mu\nu} - \tfrac{1}{4}R^2 g_{\mu\nu}).$$

In arriving at equation (3.13) we have taken the massless limit $m \to 0$ *after* renormalisation, thus avoiding an infra-red divergence. The quantity μ is an arbitrary length (or inverse mass) scale which must be introduced because the logarithmic term $^{(1)}H_{\mu\nu}\ln(\frac{1}{6}Rm^{-2})$ which arises from equation (3.12) may be arbitrarily decomposed as follows

$$^{(1)}H_{\mu\nu}\ln(\tfrac{1}{6}Rm^{-2}) = {}^{(1)}H_{\mu\nu}\ln R\mu^{-2} + {}^{(1)}H_{\mu\nu}\ln(\tfrac{1}{6}\mu^2 m^2). \tag{3.14}$$

Now in a full gravitational dynamical theory, there would be a term $^{(1)}H_{\mu\nu}$ on the left-hand side of the gravitational field equations which arises from the presence of an R^2 term in the generalised gravitational action (see I, § 3). The final term of (3.14) is proportional to $^{(1)}H_{\mu\nu}$, so it may be taken over to the left-hand side of the field equations and absorbed in the renormalisation of the coupling constant of this term. The other term on the right of equation (3.14) cannot be so absorbed, and indeed, its presence in $\langle T_{\mu\nu}\rangle_{\rm ren}$ is essential to the conservation of that quantity. An arbitrary length scale also arises in dimensional regularisation, where it has been referred to as the renormalisation point.

Equation (3.13) may be rearranged as follows:

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \langle T_{\mu\nu}\rangle_{\rm conformal} - \frac{1}{1152\pi^2}\,{}^{(1)}H_{\mu\nu}\ln R\mu^{-2} + \frac{1}{13824\pi^2}(-32R_{;\mu\nu} + 56\Box R g_{\mu\nu} - 8RR_{\mu\nu} + 11R^2 g_{\mu\nu}) \tag{3.15}$$

where $\langle T_{\mu\nu}\rangle_{\rm conformal}$ was found in I to be

$$\langle T_{\mu\nu}\rangle_{\rm conformal} = \frac{1}{2880\pi^2}({}^{(3)}H_{\mu\nu} - \tfrac{1}{6}{}^{(1)}H_{\mu\nu})$$

with $^{(3)}H_{\mu\nu} = -R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \frac{1}{12}R^2 g_{\mu\nu}$. Note that both $\langle T_{\mu\nu}\rangle_{\rm conformal}$ and the non-conformal correction terms are separately conserved, as in the two-dimensional case, a property which may be verified by direct differentiation of (3.15) using (3.6). The correction terms manifestly vanish for $R = 0$, as expected.

Taking the trace of (3.13) yields

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{1}{192\pi^2}(\Box R)\ln R + \frac{1}{17280\pi^2}[246\Box R - 6R^{\alpha\beta}R_{\alpha\beta} + (47 - 90\ln\mu^2)R^2]. \tag{3.16}$$

The *operator* $T_{\mu\nu}$ has the trace (for $\xi = 0$)

$$T^{\sigma}{}_{\sigma} = -\phi^{;\sigma}\phi_{;\sigma} - 2m^2\phi^2. \tag{3.17}$$

The first term remains even when $m = 0$. It contributes the term

$$\frac{1}{192\pi^2}\Box R \ln R + \frac{1}{17280\pi^2}\left[270\Box R - 10R^{\alpha\beta}R_{\alpha\beta} + (\tfrac{115}{3} - 90 \ln \mu^2)R^2\right] \tag{3.18}$$

to the renormalised trace (3.14). The remainder of the trace

$$(1440\pi^2)^{-1}(R^{\alpha\beta}R_{\alpha\beta} - 6\Box R + \tfrac{13}{6}R^2) \tag{3.19}$$

is 'anomalous'. It arises because G_{ren} as given by equation (3.12) contains a term proportional to m^{-2} which combines with the $2m^2$ in (3.17) to give (3.19). Thus, the mass-dependent part of the stress tensor contributes a finite trace term to the *renormalized* stress tensor even in the limit $m \to 0$. This is the famous conformal anomaly (equal to $(8\pi^2)^{-1}(1-3\xi)a_2(\xi)$ in the notation of Christensen 1976).

We come now to the second example for which an exact solution of (3.1) exists:

$$\begin{aligned} a(t) &= \sigma t^c \\ C(\eta) &= \sigma^{2/(1-c)}(1-c)^{2c/(1-c)}\eta^{2c/(1-c)} \end{aligned} \qquad (\sigma, c \text{ constants}) \tag{3.20}$$

which has been discussed by Ford and Parker (1977) in connection with infra-red divergences. For convenience we use their notation. One has mode solutions of the form

$$(2\pi)^{-3/2}\psi_k(\eta)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}} \tag{3.21}$$

where

$$\psi_k(\eta) = \left(\frac{|b|\eta}{a_0^2}\right)^{1/2b}(c_1 H_\nu^{(1)}(k\eta) + c_2 H_\nu^{(2)}(k\eta)). \tag{3.22}$$

$H^{(1)}$ and $H^{(2)}$ are Hankel functions, $k = |\boldsymbol{k}|$ and

$$b = (1-c)/(1-3c) \tag{3.23}$$

$$\nu = (2|b|)^{-1} \tag{3.24}$$

$$a_0 = \sigma[\sigma^3(1-3c)]^{c/(1-3c)}. \tag{3.25}$$

The coefficients c_1 and c_2 are complex numbers satisfying the Wronskian condition

$$|c_2|^2 - |c_1|^2 = \pi/(4b). \tag{3.26}$$

We study the vacuum state defined by $c_1 = 0$, so that in the limit $c \to 0 (\nu \to \frac{1}{2})$ we recover the standard vacuum state of Minkowski space associated with the modes $k^{-1/2} \exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{x} - \mathrm{i}kt)$.

The two-point function

$$G(x'', x') \equiv \langle\phi(x'')\phi(x')\rangle = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\boldsymbol{k}.(\boldsymbol{x}''-\boldsymbol{x}')}\psi_k(\eta'')\psi_k^*(\eta')\,\mathrm{d}^3k \tag{3.27}$$

may be evaluated by first performing the angular integration to yield an integral of the form

$$\int_0^\infty k \sin k|\boldsymbol{x}''-\boldsymbol{x}'| H_\nu^{(2)}(k\eta'')H_\nu^{(2)*}(k\eta')\,\mathrm{d}k \tag{3.28}$$

which may be expressed in terms of an associated Legendre function, or alternatively a hypergeometric function F. The result is, after some work

$$G(x'', x') = (16\pi^2\eta''\eta')^{-1}C^{-1/2}(\eta'')C^{-1/2}(\eta')$$
$$\times\left(\frac{1}{4}-\nu^2\right)\sec\pi\nu F\left(\frac{3}{2}+\nu, \frac{3}{2}-\nu; 2; 1+\frac{\Delta u\Delta v}{4\eta''\eta'}\right), \tag{3.29}$$

where once again we have chosen $|\boldsymbol{x}''-\boldsymbol{x}'| = \Delta z$, without loss of generality. The scalar curvature R is here given by

$$R = 6c(2c-1)[\sigma(1-c)\eta]^{-2/(1-c)}. \tag{3.30}$$

Curiously, the result (3.29) is closely similar to that for de Sitter space (see II, equation (3.11)) although the index ν has quite a different meaning in that case. However, we may still use the expansion given in II for the hypergeometric function to obtain, after symmetrisation over x'' and x'

$$G(x'', x') = -\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{4\pi^2\Delta u\Delta v}$$
$$+\frac{(\frac{1}{4}-\nu^2)}{16\pi^2\eta''\eta'C^{1/2}(\eta'')C^{1/2}(\eta')}\left[\ln\left(\frac{\Delta u\,\Delta v}{4\eta''\eta'}\right)+\psi\left(\frac{3}{2}+\nu\right)\right.$$
$$\left.+\psi\left(\frac{3}{2}-\nu\right)+2\gamma-1\right]-\frac{(\frac{1}{4}-\nu^2)(\frac{9}{4}-\nu^2)\Delta u\,\Delta v}{128\pi^2\eta''^2\eta'^2C^{1/2}(\eta'')C^{1/2}(\eta')}\left[\ln\left(\frac{\Delta u\,\Delta v}{4\eta''\eta'}\right)\right.$$
$$\left.+\psi\left(\frac{5}{2}+\nu\right)+\psi\left(\frac{5}{2}-\nu\right)+2\gamma-\frac{5}{2}\right]+\dots \tag{3.31}$$

where ψ here denotes the ψ function (Abramowitz and Stegun 1965).

Using the relations

$$\tfrac{1}{4}-\nu^2 = -\tfrac{1}{6}RC^2\eta^2 \tag{3.32}$$

$$\psi\left(\frac{5}{2}+\nu\right)+\psi\left(\frac{5}{2}-\nu\right) = \frac{3}{(\frac{9}{4}-\nu^2)}+\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right) \tag{3.33}$$

and expanding Δu, Δv, η, etc in powers of ϵ and t^σ, one obtains, after some tedious manipulation (real part understood)

$$G(x'', x) = -\frac{C^{-1/2}(\eta'')C^{-1/2}(\eta')}{4\pi^2\,\Delta u\,\Delta v}$$
$$+\frac{1}{576\pi^2}[-12R+\epsilon^2\Sigma(4RR_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1}-2R_{;\alpha\beta}t^\alpha t^\beta\Sigma^{-1}+2\Box R-R^2)]$$
$$\times\left[\frac{1}{2}\ln\left(\frac{\epsilon^2\Sigma}{C\eta^2}\right)+\gamma+\frac{1}{2}\psi\left(\frac{3}{2}+\nu\right)+\frac{1}{2}\psi\left(\frac{3}{2}-\nu\right)\right]+\frac{R}{96\pi^2}$$
$$-\frac{R\epsilon^2\Sigma}{288\pi^2}\left[R_{\alpha\beta}t^\alpha t^\beta\Sigma^{-1}-\frac{19}{24}R+\frac{3}{C\eta^2}\right]+\mathrm{O}(\epsilon^4). \tag{3.34}$$

Comparison of (3.34) with (3.10) and (3.11) shows that the result here is identical with the former case except for (i) the ψ terms (ii) the final η^{-2} term in (3.34). Both these terms are non-geometrical, although they are still pseudo-local in the terminology of I. Thus, the geometrical part of G is the same in both cases.

The renormalised stress tensor is found to be

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \text{equation (3.13)} + \Pi_{\mu\nu} \tag{3.35}$$

where $\Pi_{\mu\nu}$ is a non-geometrical tensor:

$$\Pi_{\mu\nu} = -\frac{{}^{(1)}H_{\mu\nu}}{1152\pi^2}\left[\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right)\right] - \frac{Rg_{\mu\nu}}{192\pi^2 C\eta^2} \tag{3.36}$$

(recall that ν depends on the geometry through (3.32)). Once again the answer is of the form $\langle T_{\mu\nu}\rangle_{\text{conformal}}$ + (correction terms which vanish when $R = 0$). By direct differentiation of (3.35) it can be established that both the conformal and non-conformal pieces of (3.36) are separately conserved, the former in any Robertson–Walker space-time, the latter when condition (3.20) is used. (Note that with the metric (3.20), equation (3.13) does *not* give a conserved tensor; hence the need for $\Pi_{\mu\nu}$.)

4. Discussion and generalisation

Although we have restricted the treatment to soluble models, some general features emerge when the structure of the regularisation and renormalisation procedure is analysed carefully.

First consider the two-dimensional case. The divergence in G is logarithmic, and in the absence of a mass the only dimensionless geometrical combination which can appear (the divergent terms must be geometrical) is $\ln \epsilon^2 R = \ln \epsilon^2 + \ln R$. The first term on the right of this decomposition gives rise to the usual quadratic divergence in $\langle T_{\mu\nu}\rangle$, but the second term, when differentiated, produces the contribution $\xi\Lambda_{\mu\nu}$. Thus, in the conformal case $\xi = 0$, the $\ln R$ term is irrelevant. In addition, the conformal anomaly term, $(1/8\pi)(\frac{1}{6}-\xi)R$, arises from a term proportional to m^{-2} in the DeWitt–Schwinger expansion, and will always be present (it is in any case uniquely determined if both the non-conformal and conformal pieces of $\langle T_{\mu\nu}\rangle_{\text{ren}}$ are to be conserved).

From these considerations it is clear that, for any quantum state and any two-dimensional space–time, the renormalised stress tensor is of the form (2.19) plus possibly a geometrical or non-geometrical, separately conserved, finite tensor which vanishes when $\xi = 0$ and when $R = 0$. (There can never be a logarithmic term in the massless renormalised stress tensor because the logarithmic divergence—which is independent of the quantum state—vanishes in two dimensions when $m = 0$; see II.)

Coming now to the four-dimensional case, we once again notice that the answer in a general Robertson–Walker space–time must consist of a common geometrical piece, given by part of (3.13) or (3.15), plus a geometrical or possibly non-geometrical term. The common geometrical piece arises (i) from the anomaly (ii) from the logarithmic terms. The logarithmic terms arise from the logarithmic divergence, and this is the same for any space–time and any state. Also, by good fortune, it is independent of t^σ and necessarily geometrical (see Christensen 1976). Hence it may be written down once and for all for a general space–time. Now, contribution (i) gives the entire answer in the conformally coupled case, $\xi = \frac{1}{6}$; the logarithmic terms are absent.

Hence the answer will always consist of a stress tensor with the form

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \langle T_{\mu\nu}\rangle_{\text{conformal}} - \frac{1}{1152\pi^2}{}^{(1)}H_{\mu\nu}\ln R\mu^{-2}$$

$$+\left(\xi-\frac{1}{6}\right)[aR_{;\mu\nu}+bRR_{\mu\nu}+c\Box Rg_{\mu\nu}+dR^2g_{\mu\nu}+\Gamma_{\mu\nu}] \quad (4.1)$$

where $\Gamma_{\mu\nu}$ is a non-geometrical tensor which vanishes when $R=0$, and a, b, c and d are coefficients to be determined. (Note that $R_{\alpha\beta}R^{\alpha\beta}$ and ${}^{(3)}H_{\mu\nu}$ cannot appear except in the conformal terms, as it does not in general vanish when $R=0$.) The last term of (4.1) is state-dependent as well as non-geometrical. It is also generally not even pseudo-local as may be seen by imposing the conservation condition on the non-conformal part of (4.1). The logarithmic term yields a $\mu=0$ component

$${}^{(1)}H_0{}^{\nu}(\ln R\mu^{-2})_{;\nu} = \frac{9}{8}C^{-2}\frac{[-8\ddot{D}D+4\dot{D}^2+3D^4][2\ddot{D}-D^3]}{(2\dot{D}+D^2)} \quad (4.2)$$

which has to cancel with a piece of $\Gamma^{\mu\nu}{}_{;\nu}$ for conservation. In the two special cases considered in § 3, one has $\ddot{D}D=2\dot{D}^2$ and $\ddot{D}=\dot{D}D$ respectively, and for these special cases the numerator of (4.2) contains a factor $2\dot{D}+D^2$. Thus, a cancellation occurs in both cases to give an accidently pseudo-local answer; in general this will not happen. Moreover, in the former case, it so happens that all pseudo-local tensors are geometrical, thus rendering the answer (3.13) 'accidently' geometrical.

The important general feature to emerge from this work is that the non-conformal answers (and this applies also to the massive case) consist of a common geometrical piece, which can be written down once and for all, plus another term, generally non-local, consisting of an integral which in general cannot be evaluated in terms of known functions, because the mode solutions of the wave equation cannot generally be written down in closed form. In spite of this, the non-local integral will be finite, and should be expressible in terms of the finite difference of two mode sums, one of which *can* be evaluated explicitly. As the result is finite, the points x'', x' can be allowed to coincide *before* integration, thus simplifying the calculation enormously.

One way in which this can be done is to use adiabatic regularisation for this 'non-point-split' piece. Although restricted to certain special space–times (Parker and Fulling 1974) Robertson–Walker is included in these, and most of the ground work has been done. A simple application (Bunch 1978) to the conformal case immediately yields the answer found by Davies *et al* (1977), but in a few lines, as a finite difference between two non-point-split, individually divergent, mode integrals. In this easy case, both mode integrals are readily evaluated, but in general only the adiabatic piece to be subtracted can be given explicitly.

We conclude with some remarks about the DeWitt–Schwinger series and the choice of vacuum state. As always in quantum mechanics it is necessary to make a choice of quantum state in which to evaluate operators as expectation values. There is, of course, no 'right' or 'wrong' state, only more or less useful ones depending on the physical situation to be described. The question of what quantum state most accurately reflects the physical contents of the real universe, or even of a simplified model, is a profound and difficult one which we do not attempt to answer here. Instead, we are here concerned with the question of *regularisation* and *renormalisation*, and the problems associated with this do not depend crucially on which state is chosen for $\langle T_{\mu\nu}\rangle$. The reason for this is that, if two different states are chosen, they can generally be connected by a Bogolubov transformation. If we can produce a regularised expression for $\langle T_{\mu\nu}\rangle$ in one state, then the expectation value in the other may be deduced from this using the Bogolubov transformation. The answer will contain an additional piece arising from the transformation which will generally be non-geometrical and finite. Of course, it may happen that if a physically unrealistic state is chosen, such as one which corresponds to an infinite amount of matter piling up on some surface of diverging blueshift (e.g. the so called Boulware vacuum around a black hole (Boulware 1976)) then the difference between the two $\langle T_{\mu\nu}\rangle$ will also be divergent on this surface.

Localised, non-geometrical singularities in $\langle T_{\mu\nu}\rangle$ arising from an unrealistic choice of quantum state are to be sharply distinguished from the singular terms in $\langle T_{\mu\nu}\rangle$ which are subtracted in the regularisation procedure. The singular terms in the unrenormalised $\langle T_{\mu\nu}\rangle$ are always given by the first three terms (or two in the two-dimensional case) of the DeWitt–Schwinger series as written out explicitly by Christensen (1976). These terms are *independent* of the choice of state; they are purely local, *geometrical* terms which are not sensitive to the global structure of the system, and in particular do *not* depend on the field boundary conditions or the choice of coordinate system in which the field modes simplify. This is easily understood; these terms represent the high-frequency, short-distance behaviour of the field, for which only the local geometry is important. However, the boundary conditions—which are closely related to the choice of quantum state—*will* affect the *finite* terms of $\langle T_{\mu\nu}\rangle$, and these state-dependent terms are precisely the (generally) non-local, non-geometrical terms such as $\Pi_{\mu\nu}$ in (3.36). Had we chosen a different quantum state, these terms would be different also. Notice, though, that we do not use the finite or higher-order terms of the DeWitt–Schwinger expansion, so we do not need to be specific about the quantum state or boundary conditions used there. In particular, they need not be the same as we use for $\langle T_{\mu\nu}\rangle$. It is most important to realise that the DeWitt–Schwinger series is *not* used to generate $\langle T_{\mu\nu}\rangle$. It is only used to calculate the divergent terms which are *subtracted off* from $\langle T_{\mu\nu}\rangle$.

A good concrete illustration of these remarks is the Casimir effect, in which the field modes are constrained by boundary conditions to be discrete. The divergent terms of $\langle T_{\mu\nu}\rangle$ are *unaffected*, and continue to be given correctly by the first terms of the (geometrical) DeWitt–Schwinger series, but the effect of mode discreteness alters the finite term in a well known (and non-geometrical) way, and thus changes the *renormalised* $\langle T_{\mu\nu}\rangle$, giving rise to an altered vacuum energy.

In this paper, the choice of vacuum state is dictated by mathematical simplicity. The Fock space associated with the modes that we explicitly write down (such as (3.22)) is the most convenient choice, but the regularisation method discussed here in no way depends on this simplyfying choice. Some other state would do equally as well. Thus there is an inevitable, and required, ambiguity in the choice of vacuum. However, once the problems of principle concerning regularisation are solved, a change in choice of state is merely a computational exercise. One may then treat, unhindered by mathematical divergences, the difficult *physical* question of which quantum state is the most appropriate for any given physical situation.

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